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Differential filters for the large eddy numerical simulation of turbulent flows

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Differential filters for the large eddy numerical simulation of turbulent flows are defined and their properties are discussed. Their main advantages are that the correlations can be expressed exactly in the so-called resolvable scale and the attenuation of the filtered function in the Fourier space can be carefully controlled.

In the large eddy simulation of turbulent flow the mean value $\bar{f}$ of a function f is defined by means of an integral normalized filter[1]

$$\bar{f}(\mathbf{x},t;a)=\int G(\mathbf{x}-\mathbf{x}';a)\,f(\mathbf{x}',t)\,d^3\mathbf{x}', \tag{1}$$

where G is the kernel of the filter and a is a characteristic length. The properties of these filters are

$$\overline{hf+kg}=h\bar{f}+k\,\bar{g},\quad \overline{\frac{\partial f}{\partial x_i}}=\frac{\partial \bar{f}}{\partial x_i},\quad \overline{\frac{\partial f}{\partial t}}=\frac{\partial \bar{f}}{\partial t}, \tag{2}$$

where f and g are two generic functions, h and k two constants, and $\bar{f}=f$ if $f=$ const, so that the filtered Navier–Stokes equations assume the form

$$\frac{\partial \bar{u}_k}{\partial x_k}=0,\quad \frac{\partial \bar{u}_i}{\partial t}+\frac{\partial\, \overline{u_i\,u_k}}{\partial x_k}=-\frac{\partial \bar{p}}{\partial x_i}+\nu\nabla^2\bar{u}_i, \tag{3}$$

and the crucial problem is to model the term $\overline{u_iu_k}$. A favored integral filter is the Gaussian one,[2,3] with the kernel G given by

$$G(\mathbf{x}-\mathbf{x}';a)=\left(\frac{6}{\pi a^2}\right)^{3/2}\exp\left(-\frac{6(\mathbf{x}-\mathbf{x}')^2}{a^2}\right), \tag{4}$$

and if we introduce a system of polar coordinates r, ϑ, ϕ centered in $\mathbf{x}$, we can write

$$\bar{f}(\mathbf{x},t;a)=\left(\frac{6}{\pi a^2}\right)^{3/2}\int \exp\left(-\frac{6r^2}{a^2}\right)f(r,\vartheta,\phi,t)\times r^2\sin\vartheta\,dr\,d\vartheta\,d\phi, \tag{5}$$

where $r^2=(\mathbf{x}-\mathbf{x}')^2$.

In this letter we will analyze the properties of an integral filter given by the expression

$$\bar{f}(\mathbf{x},t;a)=\frac{1}{4\pi a^2}\int \exp\left(-\frac{r}{a}\right)f(r,\vartheta,\phi,t)\,r\sin\vartheta\,dr\,d\vartheta\,d\phi, \tag{6}$$

with a kernel that has the form

$$G(\mathbf{x}-\mathbf{x}';a)=\frac{1}{4\pi a^2}\,\frac{\exp(-|\mathbf{x}-\mathbf{x}'|/a)}{|\mathbf{x}-\mathbf{x}'|}, \tag{7}$$

and we notice that the singularity is completely removed by the integration. This filter is interesting because it is easily shown that (6) is a particular solution[4] of the partial differential equation

$$f=\bar{f}-a^2\nabla^2\bar{f} \tag{8}$$

and the kernel (7) is the related Green's function extended to an unbounded domain, so that the differential operator (8) can be interpreted as the inverse of the integral operator (6).

By means of this inverse differential operator we can express exactly the correlation term in the so-called resolvable scale. In fact, we can write, following (8):

$$u_iu_k=(\bar{u}_i-a^2\nabla^2\bar{u}_i)(\bar{u}_k-a^2\nabla^2\bar{u}_k), \tag{9}$$

and considering that

$$\bar{u}_i\bar{u}_k=\overline{\bar{u}_i\bar{u}_k}-a^2\nabla^2\,\overline{\bar{u}_i\bar{u}_k},$$

we have, in terms of $\tau_{ik}=\overline{u_iu_k}-\bar{u}_i\bar{u}_k$,

$$\tau_{ik}=\overline{2a^2\nabla\bar{u}_i\cdot\nabla\bar{u}_k+a^4(\nabla^2\bar{u}_i)(\nabla^2\bar{u}_k)}. \tag{10}$$

We emphasize the fact that (10) represents an exact expression coherent with the filter (7) and obtained without new assumptions, and it is interesting to notice that in first approximation we have

$$\tau_{ik}\cong 2a^2\nabla\bar{u}_i\cdot\nabla\bar{u}_k, \tag{11}$$

that is the Clark *et al.*[5] model for the Leonard terms plus the cross terms.

In order to appreciate the analogies between the filter (7) and the Gaussian filter (4) we notice that the Gaussian filter gives a filtered function $\bar{f}$ such that

$$\frac{\partial \bar{f}}{\partial \epsilon}=\nabla^2\bar{f}, \tag{12}$$

where $\epsilon=a^2/24$, so that assuming

$$\frac{\partial \bar{f}}{\partial \epsilon}\cong\frac{\bar{f}-f}{\epsilon}, \tag{13}$$

we have in first approximation a differential relation between f and $\bar{f}$ similar to (8)

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$$f=\bar{f}-\epsilon\nabla^2\bar{f}. \tag{14}$$

We stress the fact that $\bar{f}$ must be considered a function of ϵ, so that when $\epsilon=0, \bar{f}=f$, and we notice that the Gaussian filter corresponds in some sense to a diffusive process of the original function f in ϵ. Finally, considering the transfer function in Fourier space $\mathbf{K}$ we have for the Gaussian filter

$$\bar{f}(\mathbf{K},t;\epsilon)=f(\mathbf{K},t)\exp(-\epsilon|\mathbf{K}|^2), \tag{15}$$

and for the filter expressed by (7)

$$\bar{f}(\mathbf{K},t;a)=\frac{f(\mathbf{K},t)}{1+a^2|\mathbf{K}|^2}, \tag{16}$$

as can be calculated directly from the differential relation (8), so that the two transfer functions become equivalent in the limit $a\to 0$.

The results obtained can be generalized in various forms, including a larger class of differential filters (that is, linear integral filters whose inverse is exactly given by a differential one). First, note that the one-dimensional version of filter (7) is given by the expression

$$G(x-x';a)=\frac{1}{2a}\exp\left(-\frac{|x-x'|}{a}\right), \tag{17}$$

so that the mean value $\bar{f}$

$$\bar{f}(x,t;a)=\frac{1}{2a}\int_{-\infty}^{\infty}\exp\left(-\frac{|x-x'|}{a}\right)f(x',t)dx \tag{18}$$

is a particular solution of the ordinary differential equation

$$f=\bar{f}-a^2\frac{d^2\bar{f}}{dx^2}. \tag{19}$$

Also note that the results can be easily extended to anisotropic filters in which f and $\bar{f}$ are related by the differential elliptic expression

$$f=\bar{f}-\alpha_{ij}\frac{\partial^2\bar{f}}{\partial x_i\,\partial x_j}, \tag{20}$$

where α_{ij} are constant terms and the quadratic form $\alpha_{ij}x_ix_j$ is always positive. In fact we can always reduce this expression to the canonic form (8) with a linear change of coordinates,[4] and then we can assume as a solution the same integral filter (7). In this case we have

$$\tau_{ij}=\overline{2\alpha_{kl}\frac{\partial\bar{u}_i}{\partial x_k}\frac{\partial\bar{u}_j}{\partial x_l}+\alpha_{kl}\alpha_{rs}\frac{\partial^2\bar{u}_i}{\partial x_k\,\partial x_l}\frac{\partial^2\bar{u}_j}{\partial x_r\,\partial x_s}}. \tag{21}$$

It is interesting to see that the transfer function can be carefully controlled in the wave space $\mathbf{K}$ by changes in the coefficients α_{ij}

$$\bar{f}(\mathbf{K},t;\alpha_{ij})=\frac{f(\mathbf{K},t)}{1+\alpha_{ij}K_iK_j}, \tag{22}$$

where we stress the fact that there is always an attenuation of the filtered function $\bar{f}$, owing to the ellipticity of the differential operator (20).

In this Letter it has not been necessary to consider explicitly the fluctuations $f'=f-\bar{f}$. However, it is interesting to interpret the results in terms of the well-known Leonard stresses

$$L_{kl}=\overline{\bar{u}_k\bar{u}_l}-\bar{u}_k\bar{u}_l,$$

the subgrid-scale cross stresses

$$C_{kl}=\overline{u'_k\bar{u}_l+\bar{u}_ku'_l},$$

and the subgrid-scale Reynolds stresses

$$R_{kl}=\overline{u'_ku'_l}.$$

In the case of the isotropic filter (7) we have in particular

$$\begin{aligned} u'_i &= -a^2\nabla^2\bar{u}_i, \quad L_{kl}=a^2\nabla^2(\overline{\bar{u}_i\bar{u}_k}),\\ C_{kl} &= -\overline{a^2(\bar{u}_k\nabla^2\bar{u}_l+\bar{u}_l\nabla^2\bar{u}_k)},\\ R_{kl} &= \overline{a^4(\nabla^2\bar{u}_k)(\nabla^2\bar{u}_l)}, \end{aligned} \tag{23}$$

and we notice again the strong analogy of these terms with previous approximate evaluations.[5]

In conclusion we think that it would be interesting to test numerically differential filters given by the expression (8) or by the more general form (20) in the case of nonhomogeneous turbulence. In addition we mention that historically the idea of filtering functions by means of differential filters is not new in turbulence studies. Kampé de Fériet and Betchov[6] analyzed in the past functions filtered in time by the low-pass integral filter

$$\bar{f}(\mathbf{x},t;\tau)=\frac{1}{\tau}\int_{-\infty}^{t}\exp\left(-\frac{t-t'}{\tau}\right)f(\mathbf{x},t')dt', \tag{24}$$

particular solution of the differential equation

$$f=\bar{f}+\tau\frac{d\bar{f}}{dt}, \tag{25}$$

but to the knowledge of the author they have never been used in modeling turbulent stresses. Finally, the author is indebted to the referee for two important observations that better clarify the limits of this work and stimulate future research. The first one regards the fact that formally the use of the filters proposed by the author is equivalent to a direct simulation with no model but with different dependent variables. In the filtered equations the nonlinear terms will still cause a loss of information; the real duty of models is to minimize this error. In the second observation the referee notes that the filters proposed by the author avoid the inconsistency between the filters used and the model forms, inconsistency that is at the origin of the disagreement between the models and the computations in Clark *et al.*,[5] but this consistency does not necessarily mean that they are useful for large eddy simulations where significant loss of information must always be modeled.

At the present stage of the research, which is in progress, it is very difficult to say if the filtered Navier–Stokes equations, that formally become integrodifferential equations, improve the numerical capturing of the large scale structure, or in other words are less chaotic than the original ones. We notice that the class of differential filters is very large, they can be extended also to the time, and potentially every Green's function associated with a differential operator in an unbounded or bounded domain can be interpreted as the kernel of an integral filter that mathematically represents the inverse operator. The author is presently engaged in the study of the properties of such filters, their adaptability to particular flows, and their algebra. Obviously their actual usefulness in large eddy simulation must be verified.

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Differential filters of elliptic type

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Linear differential filters, i.e., filters in which the filtered function $\bar f$ and the original function f are connected by a linear differential equation, are studied on a general basis concerning the elliptic operators of second order. In addition, a particular example of a parabolic filter depending on space and extended to past times is given, and its interest in the context of the large eddy simulation of turbulence is discussed.

In the large eddy simulation of turbulent flows a correspondence between the filtered function $\bar f$ and the original function f is established in the form[1]

$$\bar f(\mathbf{x},t;a) = \int G(\mathbf{x}-\mathbf{x}';a)\, f(\mathbf{x}',t)\, d^3\mathbf{x}', \tag{1}$$

where $\mathbf{x}$ is a set of Cartesian coordinates, t is the time, and a is a reference length.

In a previous article[2] it was proved the existence of linear differential filters, i.e., filters in which $\bar f$ and f are connected by a linear differential equation, that are endowed with interesting properties concerning the correlations. In this Letter a general theory of these linear differential filters concerning the second-order linear operators of the elliptic type is outlined. Let us consider the linear operator

$$M(u) = u + \alpha_i \frac{\partial u}{\partial x_i} - \alpha_{ij}\frac{\partial^2 u}{\partial x_i\, \partial x_j}, \quad \alpha_{ij} = \alpha_{ji}, \tag{2}$$

where α_i and α_{ij} are given functions of x_i and $\alpha_{ij}x_i x_j$ is always definite positive. The principal fundamental solution[3] $G(\mathbf{x},\mathbf{x}')$ of the equation $M(u)=0$ is a solution defined in the whole space, such that it satisfies the following bounds:

$$G = O(1/r), \quad r\to 0; \text{ and } G = O(e^{-r}), \quad r\to\infty,$$

where $r = |\mathbf{x}-\mathbf{x}'|$.

We notice that G has an integrable singularity when $\mathbf{x}=\mathbf{x}'$. It can be shown[3] first that Eq. (2) admits only one principal fundamental solution and secondly that for every regular function $f(\mathbf{x})$ going to infinity when $|\mathbf{x}|\to\infty$ at most like $|\mathbf{x}|^p$, with p arbitrary positive number, the following relation holds:

$$\bar f(\mathbf{x}) = \int G(\mathbf{x},\mathbf{x}') f(\mathbf{x}')\, d^3\mathbf{x}', \tag{3}$$

where

$$f = \bar f + \alpha_i \frac{\partial \bar f}{\partial x_i} - \alpha_{ij}\frac{\partial^2 \bar f}{\partial x_i\, \partial x_j}. \tag{4}$$

We notice that these properties can be interpreted as properties of a class of filters in which the kernel is the principal fundamental solution of the homogeneous equation (2) and in which the filtered function $\bar f$ and the original function f are connected not only by the integral relation (3) but also by the differential bound expressed by (4). This class of filters has the properties

$$\overline{f+g} = \bar f + \bar g, \tag{5}$$

$$\overline{kf} = k\bar f, \quad \text{if } k = \text{const}, \tag{6}$$

$$\overline{fg} = \overline{\left(\bar f + \alpha_i \frac{\partial \bar f}{\partial x_i} - \alpha_{ij}\frac{\partial^2 \bar f}{\partial x_i\, \partial x_j}\right)\left(\bar g + \alpha_i \frac{\partial \bar g}{\partial x_i} - \alpha_{ij}\frac{\partial^2 \bar g}{\partial x_i\, \partial x_j}\right)}, \tag{7}$$

where f and g are two generic functions, and we notice in addition that the kernel G is normalized so that

$$\bar f = f, \quad \text{if } f = \text{const}. \tag{8}$$

With regard to the filtered value of a generic derivative $\overline{\partial f/\partial x_k}$ we notice that $G(\mathbf{x},\mathbf{x}')$ is not necessarily a function of $\mathbf{x}-\mathbf{x}'$ and the consequence is that generally the process of derivation and the process of filtering do not commute. However, we can write, deriving Eq. (4),

$$\frac{\partial f}{\partial x_k} = \frac{\partial \bar f}{\partial x_k} + \alpha_i \frac{\partial}{\partial x_i}\left(\frac{\partial \bar f}{\partial x_k}\right) - \alpha_{ij}\frac{\partial^2}{\partial x_i\, \partial x_j}\left(\frac{\partial \bar f}{\partial x_k}\right) + \frac{\partial \alpha_i}{\partial x_k}\frac{\partial \bar f}{\partial x_i} - \frac{\partial \alpha_{ij}}{\partial x_k}\frac{\partial^2 \bar f}{\partial x_i\, \partial x_j}, \tag{9}$$

and obviously we also have

$$\frac{\partial f}{\partial x_k} = \overline{\frac{\partial f}{\partial x_k}} + \alpha_i \frac{\partial}{\partial x_i}\left(\overline{\frac{\partial f}{\partial x_k}}\right) - \alpha_{ij}\frac{\partial^2}{\partial x_i\, \partial x_j}\left(\overline{\frac{\partial f}{\partial x_k}}\right), \tag{10}$$

so that we obtain

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